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# Asymptotic Properties of Orthogonal Polynomials from Their Recurrence Formula, II

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This paper is a sequel to a previous one. It is assumed that the coefficients in the recurrence formula for orthogonal polynomials are unbounded but vary regularly and have a different behavior for even and odd indices. Asymptotics are given for the ratio of orthogonal polynomials and zero distribution is discussed. The results are then applied to orthogonal polynomials related to elliptic functions. © 1988 Academic Press, Inc.

## I. INTRODUCTION

Let $\{p_n(x)\}$ be a sequence of orthogonal polynomials on the real line with respect to a (probability) distribution function $W$:

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, dW(x) = \delta_{m,n}. \tag{1.1}$$

These polynomials satisfy a second-order recurrence relation

$$xp_n(x) = \beta_{n+1}^{1/2}\, p_{n+1}(x) + \alpha_n\, p_n(x) + \beta_n^{1/2} p_{n-1}(x), \qquad n = 0, 1, 2, ..., \tag{1.2}$$

where $p_{-1}(x) = 0$; $p_0(x) = 1$; $\alpha_n \in \mathbb{R}$; and $\beta_n > 0$. The monic polynomials

$$q_n(x) = \sqrt{\beta_1 \beta_2 \cdots \beta_n}\, p_n(x), \qquad n = 1, 2, ...,$$

can be calculated by a similar recurrence formula,

$$q_{n+1}(x) = (x - \alpha_n)\, q_n(x) - \beta_n q_{n-1}(x), \qquad n = 0, 1, 2, ..., \tag{1.3}$$

with initial values $q_{-1}(x) = 0$ and $q_0(x) = 1$. A theorem of J. Favard states that the solution of any second-order recurrence relation of the form (1.3), with $\alpha_n \in \mathbb{R}$, $\beta_n > 0$, and initial values $q_{-1}(x) = 0$ and $q_0(x) = 1$, consists of a

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sequence of polynomials that are orthogonal with respect to some (probability) distribution $W$ [10, Theorem II.1.5].

Recently a lot of work has been devoted to obtaining properties of the distribution function $W$ from the recurrence coefficients $\{\alpha_n\}$ and $\{\beta_n\}$ and vice versa. The case where these recurrence coefficients have finite limits has been treated by Chihara [4, 5], Nevai [16–18], Geronimo and Case [12], and others. Asymptotically periodic recurrence coefficients with finite accumulation points have been studied by Chihara [5] and Geronimo and the author [13, 28] (see also the references given there). In general one can say that the support of the distribution function $W$ is bounded if and only if the recurrence coefficients are bounded. If the recurrence coefficients are unbounded then the orthogonality property is extended over an unbounded set, possibly the whole real axis. Some interesting aspects of orthogonal polynomials with respect to a weight function $w(x) = W'(x)$ on $(-\infty, \infty)$ and on $(0, \infty)$ for which

$$\lim_{x \to \infty} |x|^{-\alpha} \log w(x) = -1 \qquad (\alpha > 0)$$

have been given by Rakhmanov [21], Mhaskar and Saff [15], and Ullman [26, 27; see also 22]. A well-known conjecture by Freud [11] states that the recurrence coefficients $\beta_n$ for the weight function $|x|^\rho \exp(-|x|^\alpha)$ satisfy

$$\lim_{n \to \infty} n^{-1/\alpha} \beta_n^{1/2} = \left[ \frac{\Gamma(\alpha/2)\, \Gamma(\alpha/2+1)}{\Gamma(\alpha+1)} \right]^{1/\alpha} \tag{1.4}$$

(observe that $\alpha_n = 0$ because $w$ is an even function). Freud proved this conjecture for $\alpha = 2$, 4, and 6, and a general proof for $\alpha$ an even positive integer has been given by Magnus [14]. The conjecture (1.4) indicates that the sequence $\beta_n$ increases as $n^{2/\alpha}$ so that this sequence is regularly varying with exponent $2/\alpha$. Recently the conjecture has been proved by Lubinsky, Mhaskar and Saff [29, 30]. See also [19] for more on Freud's conjecture.

Chihara [6] studies unbounded recurrence coefficients and he makes an extensive use of chain sequences to give conditions on the recurrence coefficients under which the derived set of all the zeros of the orthogonal polynomials is bounded from below (or from above). Nevai and Dehesa [20] assume that the recurrence coefficients are diverging in such a way that there exists a positive increasing function $\varphi(t)$ such that $\alpha_n/\varphi(n)$ and $\beta_n^{1/2}/\varphi(n)$ have finite limits, where this function is such that for every $x > 0$ and for $t \to \infty$, $\varphi(x+t)/\varphi(t)$ tends to one. Under these assumptions, Nevai and Dehesa give asymptotics for

$$\sum_{j=1}^{n} x_{j,n}^M \Big/ \int_0^n \varphi^M(t)\, dt,$$

where $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the zeros of $p_n(x)$ (or $q_n(x)$) and $M$ is a fixed positive integer.

In this paper we also assume that the recurence coefficients are diverging in a particular way. We will describe this divergence by means of the notion of regular variation [2, 23]:

DEFINITION. A non-negative measurable function $f: \mathbb{R}^+ \to \mathbb{R}^+$ is regularly varying (at infinity) if for some real $\alpha$ and all $t > 0$,

$$\lim_{x \to \infty} f(xt)/f(x) = t^\alpha.$$

$\alpha$ is called the exponent of regular variation.

It is clear that a regularly varying function with exponent $\alpha$ can be written as $x^\alpha L(x)$, where $L: \mathbb{R}^+ \to \mathbb{R}^+$ is a slowly varying function, i.e., for every $t > 0$,

$$\lim_{x \to \infty} L(xt)/L(x) = 1.$$

Simple examples of slowly varying functions are $|\log x|^a$, $|\log \log x|^b$, etc. If $f(x)$ is a regularly varying function (with exponent $\alpha$), then $\lambda_n = f(n)$ will be a regularly varying sequence with exponent $\alpha$.

In Section III we will suppose that there exists a regularly varying sequence $\lambda_n$ with exponent $\alpha > 0$ such that the recurrence coefficients satisfy

$$\begin{aligned} &\lim_{n \to \infty} \alpha_{2n}/\lambda_{2n} = a_1; &\qquad &\lim_{n \to \infty} \beta_{2n}^{1/2}/\lambda_{2n} = b_1 \\ &\lim_{n \to \infty} \alpha_{2n+1}/\lambda_{2n} = a_2; &\qquad &\lim_{n \to \infty} \beta_{2n+1}^{1/2}/\lambda_{2n} = b_2 \end{aligned} \tag{1.5}$$

and we will give some asymptotic properties of the related orthogonal polynomials. In Section IV we will consider the cases where $a_1 = a_2$ and $b_1 = b_2$ and the special case where $b_1$ or $b_2$ is equal to zero. In Section V we apply the results to some families of orthogonal polynomials that are related to elliptic functions.

## II. Preliminary Results

We will recall some results from the theory of orthogonal polynomials that will be used in this paper. The zeros of orthogonal polynomials are all real and simple and we will denote the zeros of $q_n$ (or $p_n$) in increasing order by $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$. An interesting upper bound for these zeros is

$$|x_{j,k}| \leqslant \max_{0 \leqslant i \leqslant k-1} |\alpha_i| + 2 \max_{1 \leqslant i \leqslant k-1} \beta_i^{1/2} \tag{2.1}$$

[20, Formula 11]. The Christoffel numbers $\{\lambda_{j,n}; j \leqslant n, n=1, 2, ...,\}$ that belong to the orthogonal polynomials $\{p_n(x)\}$ are the unique numbers for which the Gauss–Jacobi mechanical quadrature holds:

$$\int_{-\infty}^{\infty} p(x)\, dW(x) = \sum_{j=1}^{n} \lambda_{j,n}\, p(x_{j,n}),$$

where $p(x)$ is any polynomial of degree at most $2n-1$ [25, p. 47]. These numbers are given by

$$\lambda_{j,n} = \beta_n^{-1/2} \frac{1}{p_{n-1}(x_{j,n})\, p_n'(x_{j,n})} > 0$$

and one easily finds

$$\frac{q_{n-1}(x)}{q_n(x)} = \sum_{j=1}^{n} \frac{\lambda_{j,n}\, p_{n-1}^2(x_{j,n})}{x - x_{j,n}}. \tag{2.2}$$

Another interesting rational function is

$$\frac{q_n'(x)}{q_n(x)} = \sum_{j=1}^{n} \frac{1}{x - x_{j,n}}. \tag{2.3}$$

The following lemma is an important observation first made by Dombrowski [7–9]:

LEMMA 1. *If* $\{u_n\}$ *satisfies a recurrence relation*

$$u_{n+1} + a_n u_n + b_n u_{n-1} = 0 \tag{2.4}$$

*with* $u_{-1} = 0$, *then*

$$\frac{u_k^2 - u_{k+1} u_{k-1}}{b_1 b_2 \cdots b_k} = u_0^2 + \sum_{j=1}^{k} \left( \frac{a_j - a_{j-1}}{b_1 b_2 \cdots b_j} u_j u_{j-1} + \frac{b_j - b_{j-1}}{b_1 b_2 \cdots b_j} u_j u_{j-2} \right). \tag{2.5}$$

*Proof.* Let us denote $u_k^2 - u_{k+1} u_{k-1}$ by $D_k$. By using the recurrence relation (2.4) for $u_{k+1}$ one obtains

$$\begin{aligned} D_k &= u_k^2 - u_{k-1}(-a_k u_k - b_k u_{k-1}) \\ &= b_k D_{k-1} + u_k(u_k + a_k u_{k-1} + b_k u_{k-2}). \end{aligned}$$

Now use the recurrence relation (2.4) for $u_k$, then

$$D_k = b_k D_{k-1} + u_k[(a_k - a_{k-1})\, u_{k-1} + (b_k - b_{k-1})\, u_{k-2}].$$

The result (2.5) follows by iterating down this last expression. ∎

LEMMA 2. *Let $\{q_n(x)\}$ be a sequence of polynomials, then*

$$\frac{q'_{2k}(x)}{q_{2k}(x)} = -\sum_{j=1}^{k} \left(\frac{q_{2j-2}(x)}{q_{2j}(x)}\right)' \bigg/ \left(\frac{q_{2j-2}(x)}{q_{2j}(x)}\right) \tag{2.6}$$

$$\frac{q'_{2k+1}(x)}{q_{2k+1}(x)} = -\sum_{j=0}^{k} \left(\frac{q_{2j-1}(x)}{q_{2j+1}(x)}\right)' \bigg/ \left(\frac{q_{2j-1}(x)}{q_{2j+1}(x)}\right). \tag{2.7}$$

*Proof.* Equation (2.6) follows immediately from

$$\frac{q'_{2k}(x)}{q_{2k}(x)} = \sum_{j=1}^{k} \left(\frac{q'_{2j}(x)}{q_{2j}(x)} - \frac{q'_{2j-2}(x)}{q_{2j-2}(x)}\right)$$

and the formula

$$\frac{q'_{2j}(x)}{q_{2j}(x)} - \frac{q'_{2j-2}(x)}{q_{2j-2}(x)} = -\left(\frac{q_{2j-2}(x)}{q_{2j}(x)}\right)' \bigg/ \left(\frac{q_{2j-2}(x)}{q_{2j}(x)}\right).$$

Equation (2.7) follows in a similar way. ∎

We will also need the following Abelian theorem:

LEMMA 3. *Suppose $\{\varepsilon_{j,n}; j \leqslant n, n = 1, 2, ...\}$ is a bounded triangular array of complex numbers for which $\varepsilon_{j,n} \to 0$ whenever $n \to \infty$ and $j/n \to t \in [0, 1]$; then for any $z$ with $|z| < 1$,*

$$z^k \sum_{j=0}^{k} \varepsilon_{j,n} z^{-j} \to 0$$

*whenever $n \to \infty$ and $k/n \to t \in [0, 1]$.*

*Proof.* Consider a sequence $k_n$ such that $k_n/n$ tends to a fixed number $t \in [0, 1]$ when $n$ tends to infinity. By reversing the order of summation one easily finds

$$z^{k_n} \sum_{j=0}^{k_n} \varepsilon_{j,n} z^{-j} = \sum_{i=0}^{k_n} \varepsilon_{k_n - i, n} z^i.$$

For fixed $i$ we see that $\varepsilon_{k_n - i, n} \to 0$ as $n$ goes to infinity, and since

$$|\varepsilon_{k_n - i, n} z^i| \leqslant M\, |z|^i,$$

where $M$ is a positive constant, one can use Lebesgue's theorem of dominated convergence to obtain the desired result. ∎

## III. Asymptotics

By means of straightforward manipulations of the recurrence formula (1.3) one obtains [28]

$$\begin{aligned} q_{2k+2}(x) &= a_{2k}(x)\, q_{2k}(x) - b_{2k}(x)\, q_{2k-2}(x) \\ q_{2k+3}(x) &= a_{2k+1}(x)\, q_{2k+1}(x) - b_{2k+1}(x)\, q_{2k-1}(x), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} a_k(x) &= (x-\alpha_k)(x-\alpha_{k+1}) - \beta_{k+1} - \beta_k \frac{x-\alpha_{k+1}}{x-\alpha_{k-1}} \\ b_k(x) &= \beta_k \beta_{k-1} \frac{x-\alpha_{k+1}}{x-\alpha_{k-1}}. \end{aligned} \tag{3.2}$$

These relations are second-order recurrence relations for the orthogonal polynomials with even degree and odd degree, respectively.

Theorem 1. *Suppose that the recurrence coefficients satisfy condition* (1.5), *where $\lambda_n$ is a regularly varying sequence with exponent $\alpha > 0$. Let $A$ be a positive constant such that $|x_{j,n}|/\lambda_n < A$ for every $n$ (which is possible because of* (2.1)); *then as $n$ goes to infinity and $k/n \to t \in [0, 1]$,*

$$\begin{aligned} \frac{1}{\lambda_n^2} \frac{q_k(\lambda_n z)}{q_{k-2}(\lambda_n z)} &\to Q(z, t) \\ &= \frac{1}{2} \{(z - a_1 t^\alpha)(z - a_2 t^\alpha) - (b_1^2 + b_2^2)\, t^{2\alpha} \\ &\quad + \sqrt{[(z - a_1 t^\alpha)(z - a_2 t^\alpha) - (b_1^2 + b_2^2)\, t^{2\alpha}]^2 - 4 b_1^2 b_2^2 t^{4\alpha}}\} \end{aligned} \tag{3.3}$$

*uniformly for $z$ on compact subsets of $\mathbb{C} \setminus [-A, A]$.*

*Proof.* Let $\{k_n\}$ be a sequence of integers such that $k_n/n \to t$ as $n$ goes to infinity. We will first prove that the ratio

$$Q_{k_n, n}(z) = \lambda_{2n}^2 \frac{q_{2k_n-2}(\lambda_{2n} z)}{q_{2k_n}(\lambda_{2n} z)}$$

converges to $1/Q(z, t)$, where $z \in [A + d, \infty)$ with $d$ sufficiently large. By (2.2) we have, for $z \in [A + d, \infty)$,

$$\lambda_n \left| \frac{q_{k-1}(\lambda_n z)}{q_k(\lambda_n z)} \right| \leqslant \sum_{j=1}^{k} \frac{\lambda_{j,k}\, p_{k-1}^2(x_{j,k})}{z - x_{j,k}/\lambda_n} < \frac{1}{d},$$

which follows since $z - x_{j,k}/\lambda_n > z - x_{n,n}/\lambda_n > d$ (if $k \leqslant n$), hence

$$|Q_{k_n,n}(z)| < \frac{1}{d^2}. \tag{3.4}$$

Therefore there exists a subsequence $\{Q_{k_{n'},n'}(z)\}$ that converges. We will show that $Q_{k_{n'},n'}(z)$ and $Q_{k_{n'}+1,n'}(z)$ have the same limit. By Lemma 1 (with $u_k = q_{2k}(\lambda_{2n}z))$ and (3.1),

$$\begin{aligned}
&\lambda_{2n}^2 \left| \frac{q_{2k}(\lambda_{2n}z)}{q_{2k+2}(\lambda_{2n}z)} - \frac{q_{2k-2}(\lambda_{2n}z)}{q_{2k}(\lambda_{2n}z)} \right| \\
&\leqslant \lambda_{2n}^2 \left| \frac{b_2(\lambda_{2n}z)\cdots b_{2k}(\lambda_{2n}z)}{q_{2k+2}(\lambda_{2n}z)\, q_{2k}(\lambda_{2n}z)} \right| \\
&\quad + \lambda_{2n}^2 \sum_{j=0}^{k} |b_{2j+2}(\lambda_{2n}z)\cdots b_{2k}(\lambda_{2n}z)| \\
&\quad \times \left\{ |a_{2j}(\lambda_{2n}z) - a_{2j-2}(\lambda_{2n}z)| \left| \frac{q_{2j}(\lambda_{2n}z)\, q_{2j-2}(\lambda_{2n}z)}{q_{2k}(\lambda_{2n}z)\, q_{2k+2}(\lambda_{2n}z)} \right| \right. \\
&\quad \left. + |b_{2j}(\lambda_{2n}z) - b_{2j-2}(\lambda_{2n}z)| \left| \frac{q_{2j}(\lambda_{2n}z)\, q_{2j-4}(\lambda_{2n}z)}{q_{2k}(\lambda_{2n}z)\, q_{2k+2}(\lambda_{2n}z)} \right| \right\}.
\end{aligned} \tag{3.5}$$

Now by means of (3.4) one easily obtains, for $j \leqslant k$,

$$\left| \frac{q_{2j}(\lambda_{2n}z)}{q_{2k}(\lambda_{2n}z)} \right| = \prod_{i=j+1}^{k} \left| \frac{q_{2i-2}(\lambda_{2n}z)}{q_{2i}(\lambda_{2n}z)} \right| < \left( \frac{1}{\lambda_{2n}d} \right)^{2k-2j}.$$

Also one easily finds from condition (1.5) that

$$\frac{\alpha_{2k}}{\lambda_{2n}} = \frac{\alpha_{2k}}{\lambda_{2k}} \frac{\lambda_{2k}}{\lambda_{2n}} \to a_1 t^\alpha; \qquad \frac{\alpha_{2k+1}}{\lambda_{2n}} = \frac{\alpha_{2k+1}}{\lambda_{2k}} \frac{\lambda_{2k}}{\lambda_{2n}} \to a_2 t^\alpha$$

when $n \to \infty$ and $k/n \to t$, and in a similar way

$$\frac{\beta_{2k}^{1/2}}{\lambda_{2n}} \to b_1 t^\alpha; \qquad \frac{\beta_{2k+1}^{1/2}}{\lambda_{2n}} \to b_2 t^\alpha,$$

which means that for our sequence $\{k_n\}$, the array $\{b_{2j}(\lambda_{2n}z)/\lambda_{2n}^4;\ j \leqslant k_n,\ n = 1, 2, ...\}$ is bounded by a constant $C$. With this, (3.5) becomes

$$\begin{aligned}
&\lambda_{2n}^2 \left| \frac{q_{2k}(\lambda_{2n}z)}{q_{2k+2}(\lambda_{2n}z)} - \frac{q_{2k-2}(\lambda_{2n}z)}{q_{2k}(\lambda_{2n}z)} \right| \\
&\leqslant \left( \frac{C}{d^4} \right)^k + \left( \frac{C}{d^4} \right)^k \sum_{j=1}^{k} \left\{ \frac{|a_{2j}(\lambda_{2n}z) - a_{2j-2}(\lambda_{2n}z)|}{\lambda_{2n}^2 d^4} \right. \\
&\quad \left. + \frac{|b_{2j}(\lambda_{2n}z) - b_{2j-2}(\lambda_{2n}z)|}{\lambda_{2n}^4 d^6} \right\} \left( \frac{C}{d^4} \right)^{-j}.
\end{aligned}$$

Now we can choose $d$ large enough so that $C < d^4$ and by Lemma 3 it then follows that for $n \to \infty$ and $k/n \to t$,

$$Q_{k+1,n}(z) - Q_{k,n}(z) \to 0.$$

So if $\{Q_{k_{n'},n'}(z)\}$ is a subsequence that converges, then also $\{Q_{k_{n'}+1,n'}(z)\}$ converges and has the same limit. This limit can be found by using (3.1), which gives

$$\frac{1}{Q_{k_{n'}+1,n'}(z)} = \frac{a_{2k_{n'}}(\lambda_{2n'}z)}{\lambda_{2n'}^2} - \frac{b_{2k_{n'}}(\lambda_{2n'}z)}{\lambda_{2n'}^4} Q_{k_{n'},n'}(z).$$

Letting $n'$ go to infinity gives that $Q_{k_{n'},n'}(z)$ converges to $1/Q(z, t)$ for which

$$Q(z, t) = a(z, t) - b(z, t)/Q(z, t),$$

where

$$a(z, t) = (z - a_1 t^\alpha)(z - a_2 t^\alpha) - (b_1^2 + b_2^2)\, t^{2\alpha}$$

$$b(z, t) = b_1^2 b_2^2 t^{4\alpha},$$

from which it follows that

$$Q(z, t) = \tfrac{1}{2}\{a(z, t) \pm \sqrt{a^2(z, t) - 4b(z, t)}\}.$$

We have to choose the positive sign to ensure that $Q(z, t)$ goes to infinity when $z$ goes to infinity. Every converging subsequence of $\{Q_{k_n,n}(z)\}$ therefore has the same limit, from which the result follows for $z \in [A + d, \infty)$.

Now $\{Q_{k_n,n}(z)\}$ is a sequence of analytic functions on $\mathbb{C} \setminus [-A, A]$ and if we take a compact set $K$ in $\mathbb{C} \setminus [-A, A]$, then the distance $\delta$ of this compact set to the interval $[-A, A]$ is strictly positive and for $z \in K$ we have the upper bound

$$|Q_{k_n,n}(z)| \leqslant \frac{1}{\delta^2}.$$

Moreover, we know that $\{Q_{k_n,n}(z)\}$ converges on a subset of $\mathbb{C} \setminus [-A, A]$ with an accumulation point. Then it follows by the theorem of Stieltjes and Vitali [4, p. 121] that $\{Q_{k_n,n}(z)\}$ converges uniformly on compact subsets of $\mathbb{C} \setminus [-A, A]$.

One can repeat the whole argument for the sequence

$$\lambda_{2n}^2 \frac{q_{2k_n-1}(\lambda_{2n}z)}{q_{2k_n+1}(\lambda_{2n}z)}$$

and find that this sequence also converges to $1/Q(z, t)$, which will prove the theorem. ∎

*Remarks.* (a) The asymptotic formula (3.3) also holds uniformly on every compact subset of $\mathbb{C}$ that contains at most finitely many elements of $\{x_{j,k_n}/\lambda_n;\ j \leqslant k_n,\ n = 1, 2, ...,\}$ provided that any of these finite number of contracted zeros is at most a finite number of times a zero of $q_{k_n}(\lambda_n z)$.

(b) Theorem 1 also holds on the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ cut along the interval $[-A, A]$ in the sense that the ratio of the two sides of (3.3) tends to 1.

THEOREM 2. *Suppose that the recurrence coefficients satisfy condition* (1.5), *where* $\lambda_n$ *is a regularly varying sequence with exponent* $\alpha > 0$, *and let* $A$ *be a positive constant such that* $|x_{j,n}|/\lambda_n < A$ *for every* $n$, *then as* $n$ *goes to infinity and* $k/n \to t \in [0, 1]$,

$$\text{(i)} \qquad \frac{1}{\lambda_{2n}} \frac{q_{2k}(\lambda_{2n} z)}{q_{2k-1}(\lambda_{2n} z)} \to \frac{Q(z, t) + b_1^2 t^{2\alpha}}{z - a_1 t^\alpha} \tag{3.6}$$

$$\text{(ii)} \qquad \frac{1}{\lambda_{2n}} \frac{q_{2k+1}(\lambda_{2n} z)}{q_{2k}(\lambda_{2n} z)} \to \frac{Q(z, t) + b_2^2 t^{2\alpha}}{z - a_2 t^\alpha} \tag{3.7}$$

*uniformly on compact subsets of* $\mathbb{C} \backslash [-A, A]$.

*Proof.* From the recurrence formula (1.3) we obtain

$$\frac{1}{\lambda_{2n}^2} \frac{q_{2k+1}(\lambda_{2n} z)}{q_{2k-1}(\lambda_{2n} z)} = \left(z - \frac{\alpha_{2k}}{\lambda_{2n}}\right) \frac{1}{\lambda_{2n}} \frac{q_{2k}(\lambda_{2n} z)}{q_{2k-1}(\lambda_{2n} z)} - \frac{\beta_{2k}}{\lambda_{2n}^2}.$$

Let $n$ go to infinity and $k/n \to t$ and use Theorem 1, then (3.6) follows immediately. A similar reasoning holds for (3.7). ∎

THEOREM 3. *Under the same conditions as in Theorem* 1 *we have, as* $n \to \infty$,

$$\begin{aligned} \frac{1}{n} \frac{(d/dz)\, q_n(\lambda_n z)}{q_n(\lambda_n z)} &\to \frac{1}{2} \int_0^1 \frac{(\partial/\partial z)\, Q(z, t)}{Q(z, t)}\, dt \\ &= \int_0^1 \frac{(z - (a_1 + a_2)\, t^\alpha/2)\, dt}{\sqrt{[(z - a_1 t^\alpha)(z - a_2 t^\alpha) - (b_1^2 + b_2^2)\, t^{2\alpha}]^2 - 4 b_1^2 b_2^2 t^{4\alpha}}} \end{aligned} \tag{3.8}$$

*uniformly on compact subsets of* $\mathbb{C} \backslash [-A, A]$.

*Proof.* As in Theorem 1, we will first prove the result for $z \in [A + d, \infty)$. Define the function

$$g_n(t) = -\frac{d}{dz} \left( \frac{q_{2[nt]}(\lambda_{2n} z)}{q_{2[nt]+2}(\lambda_{2n} z)} \right) \Big/ \left( \frac{q_{2[nt]}(\lambda_{2n} z)}{q_{2[nt]+2}(\lambda_{2n} z)} \right), \qquad 0 \leqslant t \leqslant 1,$$

where $[a]$ is the integer part of the real number $a$. By Lemma 2 we find

$$\frac{1}{2n}\frac{(d/dz)\,q_{2n}(\lambda_{2n}z)}{q_{2n}(\lambda_{2n}z)}=\frac{1}{2}\int_0^1 g_n(t)\,dt.$$

If we fix $t$, then by Theorem 1,

$$g_n(t)\to -\frac{\partial}{\partial z}\frac{1}{Q(z,t)}\Big/\frac{1}{Q(z,t)}=\left(\frac{\partial}{\partial z}Q(z,t)\right)\Big/Q(z,t)$$

and this follows since for uniform convergence of analytic functions on compact sets it is allowed to take derivatives. Also for $t\in[(j-1)/n, j/n)$,

$$g_n(t)=-\frac{d}{dz}\left(\frac{q_{2j-2}(\lambda_{2n}z)}{q_{2j-1}(\lambda_{2n}z)}\right)\Big/\frac{q_{2j-2}(\lambda_{2n}z)}{q_{2j-1}(\lambda_{2n}z)}$$
$$-\frac{d}{dz}\left(\frac{q_{2j-1}(\lambda_{2n}z)}{q_{2j}(\lambda_{2n}z)}\right)\Big/\frac{q_{2j-1}(\lambda_{2n}z}{q_{2j}(\lambda_{2n}z)}.$$

If we use the inequality $(a_j, b_j>0, j=1, ..., n)$

$$\sum_{j=1}^{n} a_j b_j\Big/\sum_{j=1}^{n} b_j\leqslant \max_{j\leqslant n} a_j$$

then we find

$$-\frac{d}{dz}\left(\frac{q_{k-1}(\lambda_n z)}{q_k(\lambda_n z)}\right)\Big/\frac{q_{k-1}(\lambda_n z)}{q_k(\lambda_n z)}$$
$$=\sum_{j=1}^{k}\frac{\lambda_{j,k}\,p_{k-1}^2(x_{j,k})}{(z-x_{j,k}/\lambda_n)^2}\Big/\sum_{j=1}^{k}\frac{\lambda_{j,k}\,p_{k-1}^2(x_{j,k})}{z-x_{j,k}/\lambda_n}$$
$$\leqslant \max_{j\leqslant k}\frac{1}{z-x_{j,k}/\lambda_n}<\frac{1}{d},$$

then it is obvious that

$$\tfrac{1}{2}g_n(t)<1/d.$$

Lebesgue's theorem of dominated convergence then gives (3.8) for $z\in[A+d,\infty)$. The general result follows by the Stieltjes–Vitali theorem. The same argument works for odd indices. ∎

Theorems 2 and 3 may be used to construct quadrature formulas that use zeros of orthogonal polynomials. Just as in [28], we will use the concept of weak convergence. For $|\gamma|\leqslant\alpha\leqslant\beta$ we define the distribution functions

$$F(x;\alpha,\beta)=\frac{1}{\pi}\int_{-\infty}^{x}\frac{|t|}{\sqrt{\beta^2-t^2}\sqrt{t^2-\alpha^2}}\,I_B(t)\,dt$$

$$G(x;\alpha,\beta,\gamma)=\frac{2}{\pi}\frac{\{(\alpha^2-\gamma^2)^{1/2}+(\beta^2-\gamma^2)^{1/2}\}^2}{(\beta^2-\alpha^2)^2}\times\int_{-\infty}^{x}\frac{\sqrt{\beta^2-t^2}\sqrt{t^2-\alpha^2}}{|t-\gamma|}\,I_B(t)\,dt,$$

where $I_B(t)$ is the indicator function of the set $B=[-\beta,-\alpha]\cup[\alpha,\beta]$. We also use the notation

$$a_{(1)}=\min(a_1,a_2);\qquad a_{(2)}=\max(a_1,a_2)$$
$$b_{(1)}=\min(b_1,b_2);\qquad b_{(2)}=\max(b_1,b_2).$$

THEOREM 4. *Under the conditions of Theorem* 1 *we have, for every continuous function $f$ as $n\to\infty$ and $k/n\to t\in[0,1]$,*

(i)
$$\sum_{j=1}^{2k}\lambda_{j,2k}\,p_{2k-1}^2(x_{j,2k})\,f(x_{j,2k}/\lambda_{2n})\to\frac{b_{(1)}^2}{b_1^2}\int_{-\infty}^{\infty}f(x)\,dG\left(x-\frac{a_1+a_2}{2}t^\alpha;\delta t^\alpha,\beta t^\alpha,\gamma t^\alpha\right)+\frac{b_1^2-b_{(1)}^2}{b_1^2}f(a_2t^\alpha),$$

(ii)
$$\sum_{j=1}^{2k+1}\lambda_{j,2k+1}\,p_{2k}^2(x_{j,2k+1})\,f(x_{j,2k+1}/\lambda_{2n})\to\frac{b_{(1)}^2}{b_2^2}\int_{-\infty}^{\infty}f(x)\,dG\left(x-\frac{a_1+a_2}{2}t^\alpha;\delta t^\alpha,\beta t^\alpha,\gamma t^\alpha\right)+\frac{b_2^2-b_{(1)}^2}{b_2^2}f(a_1t^\alpha),$$

(iii)
$$\frac{1}{n}\sum_{j=1}^{n}f(x_{j,n}/\lambda_n)\to\int_0^1\int_{-\infty}^{\infty}f(x)\,dF\left(x-\frac{a_1+a_2}{2}t^\alpha;\delta t^\alpha,\beta t^\alpha\right)dt,$$

*where*

$$\delta^2=\left(\frac{a_1-a_2}{2}\right)^2+(b_1-b_2)^2;\qquad\beta^2=\left(\frac{a_1-a_2}{2}\right)^2+(b_1+b_2)^2$$

$$\gamma=(a_1-a_2)/2,$$

*Proof.* Introduce the discrete distribution functions

$$G_{k,n}(x) = \sum_{j=1}^{k} \lambda_{j,k}\, p_{k-1}^2(x_{j,k})\, U(x - x_{j,k}/\lambda_n)$$

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} U(x - x_{j,n}/\lambda_n),$$

where $U(x)$ is the Heaviside function

$$U(x) = 1, \qquad x \geqslant 0$$
$$= 0, \qquad x < 0.$$

The Stieltjes transforms of these distribution functions are

$$S(G_{k,n}(x); z) = \int_{-\infty}^{\infty} \frac{dG_{k,n}(x)}{z - x} = \lambda_n \frac{q_{k-1}(\lambda_n z)}{q_k(\lambda_n z)} \tag{3.9}$$

$$S(F_n(x); z) = \int_{-\infty}^{\infty} \frac{dF_n(x)}{z - x} = \frac{1}{n} \frac{(d/dz)\, q_n(\lambda_n z)}{q_n(\lambda_n z)}. \tag{3.10}$$

The asymptotic behaviour of these Stieltjes transforms is given by Theorems 2 and 3, and by a theorem of Grommer and Hamburger the limits have to be Stieltjes transforms of the weak limits of the distribution functions $G_{k,n}$ and $F_n$ [28]. The identification of the limits of (3.9) can be done by using Theorem 3 in [28]. To identify the limit of (3.10), we observe that the integrand of (3.8) is the Stieltjes transform of the distribution function $F(x - (a_1 + a_2)\, t^\alpha/2;\, \delta t^\alpha,\, \beta t^\alpha)$, from which the result follows. ∎

Theorem 4(iii) is actually a result about the distribution of the zeros of the ortohogonal polynomials in consideration. If we denote by $N(n; p, q)$ the number of zeros of $q_n(x)$ in the interval $[p, q]$, then

$$\frac{N(n; p\lambda_n, q\lambda_n)}{n} \to \int_0^1 \int_p^q dF\left(x - \frac{a_1 + a_2}{2}\, t^\alpha;\, \delta t^\alpha,\, \beta t^\alpha\right) dt.$$

The asymptotic distribution of the zeros of orthgonal polynomials with regularly varying recurrence coefficients is therefore equal to an integrated version of the asymptotic distribution of the zeros of orthogonal polynomials with bounded recurrence coefficients (with has been calculated in [28]).

## IV. Special Cases

Let us take a closer look at some special cases of the previous theorems. If $a_1 = a_2 = a$ and $b_1 = b_2 = b > 0$, then Theorem 4 becomes

$$\sum_{j=1}^{k} \lambda_{j,k} p_{k-1}^2(x_{j,k}) f(x_{j,k}/\lambda_n)$$
$$\to \frac{1}{2b^2\pi} \int_{a-2b}^{a+2b} \sqrt{4b^2-(x-a)^2}\, f(xt^\alpha)\, dx,$$
$$\frac{1}{n} \sum_{j=1}^{n} f(x_{j,n}/\lambda_n) \to \frac{1}{\pi} \int_0^1 \int_{a-2b}^{a+2b} f(xt^\alpha) \frac{dx}{\sqrt{4b^2-(x-a)^2}}\, dt.$$

If we put $f(x)=x^M$ in this last expression, then

$$\frac{1}{n} \sum_{j=1}^{n} \left(\frac{x_{j,n}}{\lambda_n}\right)^M \to \frac{1}{\pi} \frac{1}{M\alpha+1} \int_{a-2b}^{a+2b} \frac{y^M\, dy}{\sqrt{4b^2-(y-a)^2}},$$

which gives a result similar to results of Nevai and Dehesa [20].

An important case to consider is when $b_1$ or $b_2$ is equal to zero. In that case we see that $\delta=\beta$ in Theorem 4 and the distribution functions $F$ and $G$ will become degenerate at two points:

$$F(x; \beta, \beta) = \tfrac{1}{2}U(x-\beta) + \tfrac{1}{2}U(x+\beta)$$
$$G(x; \beta, \beta, \gamma) = \frac{\beta+\gamma}{2\beta} U(x-\beta) + \frac{\beta-\gamma}{2\beta} U(x+\beta).$$

Theorem 4 then becomes

$$\sum_{j=1}^{2k} \lambda_{j,2} p_{2k-1}^2(x_{j,2k}) f(x_{j,2k}/\lambda_{2n})$$
$$\begin{cases} \to f(a_2 t^\alpha) & \text{if } b_2=0 \\ \to \dfrac{1}{2\beta}\left\{\left(\dfrac{a_2-a_1}{2}+\beta\right) f\left(\left(\dfrac{a_1+a_2}{2}+\beta\right) t^\alpha\right)\right. & \\ \qquad \left. + \left(\dfrac{a_1-a_2}{2}+\beta\right) f\left(\left(\dfrac{a_1+a_2}{2}-\beta\right) t^\alpha\right)\right\} & \text{if } b_1=0 \end{cases}$$

$$\sum_{j=1}^{2k+1} \lambda_{j,2k+1} p_{2k}^2(x_{j,2k+1}) f(x_{j,2k+1}/\lambda_{2n})$$
$$\begin{cases} \to f(a_1 t^\alpha) & \text{if } b_1=0 \\ \to \dfrac{1}{2\beta}\left\{\left(\dfrac{a_1-a_2}{2}+\beta\right) f\left(\left(\dfrac{a_1+a_2}{2}+\beta\right) t^\alpha\right)\right. & \\ \qquad \left. + \left(\dfrac{a_2-a_1}{2}+\beta\right) f\left(\left(\dfrac{a_1+a_2}{2}-\beta\right) t^\alpha\right)\right\} & \text{if } b_2=0 \end{cases}$$

$$\frac{1}{n} \sum_{j=1}^{n} f(x_{j,n}/\lambda_n) \to \frac{1}{2} \int_0^1 \left\{ f\left(\left(\frac{a_1+a_2}{2}+\beta\right) t^\alpha\right) + f\left(\left(\frac{a_1+a_2}{2}-\beta\right) t^\alpha\right)\right\} dt,$$

where $f$ is a continuous function, $n$ goes to infinity, $k/n$ goes to $t$, and

$$\beta^2 = \left(\frac{a_1 - a_2}{2}\right)^2 + b_{(2)}^2.$$

## V. Examples

Important families of orthogonal polynomials satisfy a recurrence formula with regularly varying coefficients, such as the Laguerre polynomials, Hermite polynomials, Meixner polynomials (of the first and second kind), Poisson–Charlier polynomials, and Carlitz polynomials. To all of these families one can apply the results obtained in the previous sections. These polynomials have already been treated by Nevai and Dehesa [20]. We will have a closer look at polynomials with a different limit for the odd and the even indices.

Stieltjes [24], Carlitz [3], and Al-Salam [1] have studied two systems of orthogonal polynomials $\{C_n(x)\}$ and $\{D_n(x)\}$ with recurrence formulas

$$C_{n+1}(x) = xC_n(x) - \alpha_n C_{n-1}(x) \tag{5.1}$$

$$D_{n+1}(x) = xD_n(x) - \beta_n D_{n-1}(x) \tag{5.2}$$

with

$$\alpha_{2n} = (2n)^2 k^2; \qquad \alpha_{2n+1} = (2n+1)^2 \tag{5.3}$$

$$\beta_{2n} = (2n)^2; \qquad \beta_{2n+1} = (2n+1)^2 k^2 \tag{5.4}$$

and $k$ a real number (see also [4, p. 193]). These orthogonal polynomials are related to elliptic functions in the following way. Define

$$K(k^2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

and

$$q = \exp\left\{-\pi \frac{K(1-k^2)}{K(k^2)}\right\}.$$

If $0 < k < 1$ then $0 < q < 1$ and the distribution functions $W(x)$ for the polynomials $\{C_n(x)\}$ and $\{D_n(x)\}$ respectively are

$$W_C(x) = \sum_{j=-\infty}^{\infty} \frac{q^{(2j+1)/2}}{1 + q^{2j+1}} U\left(x - \frac{2j+1}{2K(k^2)} \pi\right)$$

$$W_D(x) = \sum_{j=-\infty}^{\infty} \frac{q^j}{1 + q^{2j}} U\left(x - \frac{j}{K(k^2)} \pi\right),$$

where $U(x)$ is the Heaviside function. This means that the polynomials $\{C_n(x)\}$ and $\{D_n(x)\}$ are orthogonal on a lattice with span $\pi/K(k^2)$.

It is obvious that condition (1.5) is valid in both cases with $\lambda_n = n$ ($\alpha = 1$). For the polynomials defined in (5.1) we find $a_1 = a_2 = 0$ and $b_1 = k$; $b_2 = 1$. For the polynomials given by (5.2) we have $a_1 = a_2 = 0$ and $b_1 = 1$; $b_2 = k$. If we use Theorem 1 then we obtain

$$\lim \frac{1}{n^2} \frac{C_k(nz)}{C_{k-2}(nz)}$$

$$= \lim \frac{1}{n^2} \frac{D_k(nz)}{D_{k-2}(nz)}$$

$$= \frac{1}{2} \{z^2 - (k^2+1)\,t^2 + \sqrt{[z^2 - (k-1)^2\,t^2][z^2 - (k+1)^2\,t^2]}\}$$

uniformly for $z$ on compact sets of $\mathbb{C} \setminus [-A, A]$. If we use the upperbound (2.1), then it follows that one can take $A = 2$ if $0 < k < 1$. The limit here means, of course, that $n$ goes to infinity and $k/n$ to $t \in [0, 1]$. The value 2 for $A$ is certainly not the best possible and probably the asymptotic behavior holds uniformly for $z$ on a compact subset of $\mathbb{C} \setminus ([-(1+k)\,t, -(1-k)\,t] \cup [(1-k)\,t, (1+k)\,t])$. However, it is not so easy to prove this last statement.

Let $\{c_{j,n}\}$ be the zeros of $C_n(x)$ and $\{d_{j,n}\}$ the zeros of $D_n(x)$, then by Theorem 4 we have for every continuous function $f$ and for $0 < k < 1$,

$$\lim \sum_{j=1}^{2k} \lambda_{j,2k}^{C} \hat{C}_{2k-1}^2(c_{j,2k}) f(c_{j,2k}/2n)$$

$$= \lim \sum_{j=1}^{2k+1} \lambda_{j,2k+1}^{D} \hat{D}_{2k}^2(d_{j,2k+1}) f(d_{j,2k+1}/2n)$$

$$= \frac{1}{\pi k} \int_{1-k}^{1+k} \frac{f(tx) + f(-tx)}{2} \frac{\sqrt{(1+k)^2 - x^2}\,\sqrt{x^2 - (1-k)^2}}{x}\,dx$$

$$\lim \sum_{j=1}^{2k+1} \lambda_{j,2k+1}^{C} \hat{C}_{2k}^2(c_{j,2k+1}) f(c_{j,2k+1}/2n)$$

$$= \lim \sum_{j=1}^{2k} \lambda_{j,2k}^{D} \hat{D}_{2k-1}^2(d_{j,2k}) f(d_{j,2k}/2n)$$

$$= \frac{1}{\pi} \int_{1-k}^{1+k} \frac{f(tx) + f(-tx)}{2} \frac{\sqrt{(1+k)^2 + x^2}\,\sqrt{x^2 - (1-k)^2}}{x}\,dx,$$

where the meaning of $\lambda^C_{j,n}$ and $\lambda^D_{j,n}$ is obvious and $\{\hat{C}_n(x)\}$ and $\{\hat{D}_n(x)\}$ are the normalized orthogonal polynomials. Finally, we also find

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(c_{j,n}/n) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(d_{j,n}/n)$$
$$= \frac{1}{\pi}\int_0^1\int_{1-k}^{1+k} (f(tx)+f(-tx))\frac{x}{\sqrt{(1+k)^2-x^2}\sqrt{(1-k)^2-x^2}}\,dx\,dt.$$

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